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# A HYDRODYNAMIC GENERALIZATION OF WARD'S IDENTITY*

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An important part of the construction of solutions of equations describing non-linear dynamic systems is the investigation of their symmetry properties. An example is provided by quantum electrodynamics, in which the gauge invariance condition implies a certain relationship between the mass operator and the vortex; this relationship is known as the generalized Ward identity (GWI) /1/. The GWI is essential for proving the renormalizability of quantum electrodynamics, i.e., the reduction of the divergences of the radiation corrections to the vertex function and the renormalization constant of the electron wave function. Similarly, in statistical hydrodynamics (turbulence theory) the Galilean invariance condition implies the existence of relationships between the different statistical moments and the response functions to external forces. Relationships of this kind were first obtained by Pitayevskii /2/ in a study of the superfluidity of liquid helium, but they have not been written out for developed turbulence. An attempt /3/ made to postulate a certain GWI-type relationship in order to establish a closed system of equations in turbulence theory has since been proved wrong /3, 4/.

In this paper, in connection with a hydrodynamic system described by the Navier-Stokes equations in the presence of a random external force, the Galilean invariance condition is seen to imply an exact relationship between the mass operator and the vertex (hydrodynamic GWI); this relationship is shown to be valid up to third-order perturbation theory.

*1. The initial system of equations.* The characteristics of the hydrodynamic field - the pressure $p$ and velocity projections $v_i$ $(i = 1, 2, 3)$ - will be considered here as components of a four-component vector $\psi_\alpha = \{\psi_0, \psi_i\} = \{p, v_i\}$ $(\alpha = 0, 1, 2, 3)$. The totality of space-time coordinates will be denoted by digits in accordance with the definition $\{\mathbf{r}_1, t_1\} = 1$. Using the formalism proposed in /5/, we write the Navier-Stokes equations as follows /6/:

$$-L_\alpha(1, [\psi]) + X_\alpha(1) + \eta_\alpha^*(1) = 0 \tag{1.1}$$

$$L_\alpha(1, [\psi]) = L_{\alpha\beta}^{(0)}(12)\,\psi_\beta(2) + {}^1/_2\,\lambda V_{\alpha\beta\gamma}(1\,|\,23)\,\psi_\beta(2)\,\psi_\gamma(3) \tag{1.2}$$

(formula (1.2) involves the formally introduced parameter $\lambda$ of the series expansion, which will ultimately be equated to unity). The linear part of the Navier-Stokes operator $L_{\alpha\beta}^{(0)}$ and the non-vanishing components of the tensor $V_{\alpha\beta\gamma}$ are determined by the relations

$$L_{\alpha\beta}^{(0)}(12) = \begin{Vmatrix} 0 & \partial_j^{(1)} \\ \partial_i^{(1)} & (\partial_t^{(1)} - \nu\Delta^{(1)})\,\delta_{ij} \end{Vmatrix} \delta(1-2) \tag{1.3}$$
$$V_{ijk}(1\,|\,23) = -[\delta_{ik}\partial_j^{(3)} + \delta_{ij}\partial_k^{(2)}]\,\delta(1-2)\,\delta(1-3)$$

$X_\alpha = \{X_0, X_i\}$ are the densities of the statistically defined sources of mass $(X_0)$ and force $(X_i)$; $\eta_\alpha^* = \{\eta_0^*, \eta_i^*\}$ are the densities of the corresponding regular (deterministic) sources. We shall assume that $X_\alpha$ is a random process of the Gaussian "white noise" type (Wyld's model /3/), for which the only non-zero cumulative mean has the form

$$\langle X_i(1)\,X_j(2)\rangle = B_{ij}(12) = \delta_{ij}\delta(t_1 - t_2)\,B(\mathbf{r}_1 - \mathbf{r}_2) \tag{1.4}$$

The characteristic functional of the system can be written as a double functional (continuous) integral with respect to the fields $\psi, \psi^*$ /6-8/:

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$$W[\psi, \psi^*] = \int d[\psi]\, d[\psi^*] \exp i\,\{S[\psi, \psi^*] + \eta_\alpha(1)\,\psi_\alpha(1) + \eta_\alpha^*(1)\,\psi_\alpha^*(1)\} \tag{1.5}$$
$$S[\psi, \psi^*] = -\psi_\alpha^*(1)\, L_\alpha(1, [\psi]) + {}^1\!/_2 i\psi_i^*(1)\, B_{ij}(12)\, \psi_j^*(2)$$

The invariance of the functional integral under a translation of the function argument $\psi^* \to \psi^* + \varphi^*$ implies a functional differential equation for the characteristic functional:

$$\left\{ \frac{\delta S[\psi, \psi^*]}{\delta \psi_\alpha^*(1)} \bigg|_{\psi=\delta/i\delta\eta,\ \psi^*=\delta/i\delta\eta^*} + \eta_\alpha^*(1) \right\} W[\eta, \eta^*] = 0. \tag{1.6}$$

The representation (1.5) is a convenient tool for constructing perturbation theory and the corresponding diagram technique. According to perturbation theory for statistical hydrodynamics /9/, the elements of the diagram technique are Green's functions $G^{(0)}$, the velocity pair correlator $C^{(0)}$ and the vertex $V$, whose Fourier-transform representations are as follows:

$$G_{ij}^{(0)}(\mathbf{p}, \omega) = P_{ij}(\mathbf{p})\,(-i\omega + \nu p^2)^{-1}$$
$$C_{ij}^{(0)}(\mathbf{p}, \omega) = P_{ij}(\mathbf{p})\, B(\mathbf{p})\,(\omega^2 + \nu^2 p^4)^{-1} \tag{1.7}$$
$$V_{ijk}(\mathbf{p}) = i\,(p_j \delta_{ik} + p_k \delta_{ij})$$

where $P_{ij}(\mathbf{p}) = \delta_{ij} - p_i p_j / p^2$ is the transversal projector. The appropriate graphic symbols are shown in Fig.1.

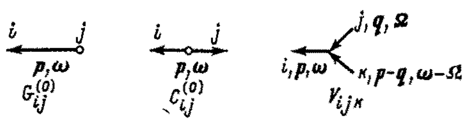


Fig.1

**2.** ***The GWI in lower-order perturbation theory.*** Consider a vertex corresponding to absorption of a quantum with zero frequency and wave number, which is the asymptotic form of an Eulerian vertex. Following /10/, we shall represent such kinematic vertices in diagrams by inserting a tailless triangular arrow, as shown in Fig.2.

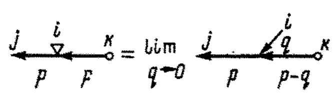


Fig.2

According to the rules of diagram technique, the expression corresponding to the diagram of Fig.2 is

$$\lambda G_{jj'}^{(0)}(\mathbf{p}, \omega)\, V_{j'ik'}(\mathbf{p})\, G_{k'k}^{(0)}(\mathbf{p}, \omega) =$$
$$i\lambda \frac{P_{jj'}(\mathbf{p})\,(p_i \delta_{j'k'} + p_{k'} \delta_{j'i})\, P_{kk'}(\mathbf{p})}{(-i\omega + \nu p^2)^2} = i\lambda p_i \frac{P_{jk}(\mathbf{p})}{(-i\omega + \nu p^2)^2} = \lambda p_i \frac{\partial G_{jk}^{(0)}(\mathbf{p}, \omega)}{\partial \omega}$$

Proceeding as in /11/, we can rewrite this expression as a GWI:

$$\lambda V_{jik}(\mathbf{p}) = -\lambda p_i \partial\, [G_{jk}^{(0)}(\mathbf{p}, \omega)]^{-1} / \partial\omega \tag{2.1}$$

Formula (2.1), which pertains to lower-order perturbation theory, cannot simply be extended to diagrams of arbitrary order by a procedure similar to that used in quantum electrodynamics /11/. The root of the difference is that in quantum electrodynamics the operation corresponding to photon insertion is multiplication by a constant Dirac matrix, whereas in hydrodynamics insertion of a kinematic vertex in an internal line is equivalent to multiplication by the momentum (wave number) of an internal line, not an external one /10/.

Nevertheless, a GWI generalizing (2.1) is valid in higher orders of perturbation theory, as will be illustrated below for third-order diagrams.

The radiation corrections to the vertex are described by a sum of three diagrams, as shown in Fig.3. Omitting the zero index of Green's functions and the correlators, we obain from Fig.3 and (1.7)

$$\Lambda^{(3)}_{jik}(p\,|\,0,\,p) = \lambda^3 V_{jmn}(\mathbf{p}) \int \frac{d^4q}{(2\pi)^4} \{G_{mm'}(q)\, G_{nn'}(p-q)\, V_{m'm''i}(\mathbf{q}) \times \tag{2.2}$$
$$C_{m''n''}(q)\, V_{n'n''k}(\mathbf{p}-\mathbf{q}) + G_{mm'}(p-q)\, C_{nn'}(q)\, V_{m'm''i}(\mathbf{p}-\mathbf{q})\, G_{m''n''}(p-q) \times$$
$$V_{n''n'k}(\mathbf{p}-\mathbf{q}) + C_{mm'}(q)\, G_{nn'}(p-q) V_{n'n''k}(\mathbf{p}-\mathbf{q})\, G_{n''m''}(-q)\, V_{m''m'i}(-\mathbf{q})\} =$$

$$i\lambda^3 \int \frac{d^4q}{(2\pi)^4} \frac{b_{jk}(\mathbf{p},\mathbf{q})\, B(\mathbf{q})}{(\Omega^2+\nu^2\mathbf{p}^4)\,[-i(\omega-\Omega)+\nu(\mathbf{p}-\mathbf{q})^2]} \times$$
$$\left\{\frac{q_i}{-i\Omega+\nu p^2} + \frac{p_i-q_i}{-i(\omega-\Omega)+\nu(\mathbf{p}-\mathbf{q})^2} - \frac{q_i}{i\Omega+\nu\mathbf{q}^2}\right\}$$
$$b_{jk}(\mathbf{p},\mathbf{q}) = V_{jmn}(\mathbf{p})\, P_{nn'}(\mathbf{q})\, P_{mm'}(\mathbf{p}-\mathbf{q})\, V_{m'n'k}(\mathbf{p}-\mathbf{q}), \quad p=\{\mathbf{p},\omega\}, \quad q=\{\mathbf{q},\Omega\}$$

(to establish this formula we have used the following properties of the projection operators: $p_i P_{lj}(\mathbf{p}) = 0$, $P_{jl}(\mathbf{p})\, P_{lk}(\mathbf{p}) = P_{jk}(\mathbf{p})$). A direct calculation shows that the sum of the terms proportional to $q_i$ in the curly brackets in (2.2) vanishes on integrating with respect to $\Omega$, and there remains

$$\Lambda^{(3)}_{jik}(p\,|\,0,\,p) = p_i \lambda \partial \Sigma^{(2)}_{jk}(p)/\partial\omega \tag{2.3}$$

where the eigenenergy operator in second-order perturbation theory is defined by

$$\Sigma^{(2)}_{jk}(p) = \lambda^2 V_{jmn}(\mathbf{p}) \int \frac{d^4q}{(2\pi)^4} C_{nn'}(q)\, G_{mm'}(p-q)\, V_{m'n'k}(\mathbf{p}-\mathbf{q}) \tag{2.4}$$

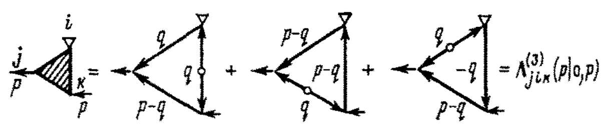


Fig.3

*3. General proof of the GWI.* We shall now show, without appealing to perturbation theory, how the GWI follows from the condition that the hydrodynamic system possesses Galilean invariance. To that end we transform formula (1.5) by applying the change of functional argument $\psi_i \to \psi_i + V_i$, which does not affect the value of the functional integral, and the coordinate transformation $x_i \to x_i - \lambda V_i t$. In view of the Galilean invariance of the Navier-Stokes operator (1.1),

$$L_l(\mathbf{r}_1 - \lambda\mathbf{V}t_1,\, t_1;\, [\psi_0, \psi_i + V_i]) = L_l(\mathbf{r}_1, t_1; [\psi_0, \psi_i]) = \tag{3.1}$$
$$-\delta S\,[\psi, \psi^*]/\delta\psi_i^*(1)$$

the functional integral (1.5) takes the form

$$W[\eta, \eta^*] = \int d[\psi]\, d[\psi^*] \exp i\,\{\psi_l^*(\mathbf{r}_1 - \lambda\mathbf{V}t_1, t_1) \times \tag{3.2}$$
$$\delta S\,[\psi, \psi^*]/\delta\psi_l^*(\mathbf{r}_1, t_1) + V_i \int d1\, \eta_i(1) + \eta_\alpha(1)\,\psi_\alpha(1) + \eta_\alpha^*(1)\,\psi_\alpha^*(1)\}$$

Since the functional integral (1.5) is independent of the parameter $V_i$ we find, using (1.6), that

$$\frac{\delta W[\eta, \eta^*]}{\delta V_i} = \int d1 \left\{\lambda t_1 \eta_l^*(1)\, \partial_i^{(1)} \frac{\delta}{i\delta\eta_l^*(1)} + \eta_i(1)\right\} W[\eta, \eta^*] = 0 \tag{3.3}$$

To obtain an equation for strongly connected (one-particle-irreducible) diagrams, we transform to new functional variables

$$\varphi_\alpha(1) = \frac{\delta}{i\delta\eta_\alpha(1)} \ln W[\eta, \eta^*], \quad \varphi_\alpha^*(1) = \frac{\delta}{i\delta\eta_\alpha^*(1)} \ln W[\eta, \eta^*]$$

after applying the Legendre functional transformation by introducing a new characteristic functional /9/

$$\Psi[\varphi, \varphi^*] = \ln W[\eta, \eta^*] - i\eta_\alpha(1)\varphi_\alpha(1) - i\eta_\alpha^*(1)\varphi_\alpha^*(1)$$

In the new variables formula (3.3) becomes

$$\int d1 \left[\lambda t_1 \partial_i^{(1)} \varphi_l^*(1) \frac{\delta\Psi}{i\delta\varphi_l^*(1)} + \frac{\delta\Psi}{i\delta\varphi_i(1)}\right] = 0 \tag{3.4}$$

This formula is a generating equation for GWI's due to Galilean invariance for strongly connected diagrams. In particular, repeated functional differentiation of (3.4) with respect to $\varphi_j^*(2), \varphi_k(3)$, using the fact that the extremality condition $\delta\Psi/i\delta\varphi = -\eta = 0$ implies $\varphi^* = 0$, yields

$$\int d1\, [\lambda t_1 G_{lk}^{-1}(13)\partial_i^{(1)}\delta_{jl}\delta(1-2) - \Gamma_{jik}(2 \mid 13)] = 0 \tag{3.5}$$

$$G_{ij}^{-1}(12) = -i\frac{\delta^2\Psi}{i\delta\varphi_i^*(1)\, i\delta\varphi_j(2)}, \quad \Gamma_{jik}(2 \mid 13) = \frac{\delta^3\Psi}{i\delta\varphi_j^*(2)\, i\delta\varphi_i(1)\, i\delta\varphi_k(3)}$$

where $G_{ij}^{-1}(12)$ is the inverse complete Green's function, and $\Gamma_{jik}(2 \mid 13)$ is a one-particle-irreducible complete vertex with two incoming lines $(1, i)$ and $(3, k)$ and one outgoing line $(2, j)$ /9/.Taking Fourier transformations of (3.5), we obtain

$$\lambda p_i \partial G_{jk}^{-1}(p)/\partial\omega + \Gamma_{jik}(p \mid 0, p) = 0 \tag{3.6}$$

This exact equation is a rigorous corollary of the Galilean invariance of the hydrodynamic system and its derivation is not based on perturbation theory.

Separating out the first-order term from the vertex

$$\Gamma_{jik}(p \mid q, p-q) = \lambda V_{jik}(p) + \Lambda_{jik}(p \mid q, p-q)$$

and using the relation

$$G_{ij}^{-1}(p) = [G_{ij}^{(0)}(p)]^{-1} - \Sigma_{ij}(p)$$

which follows from the Dyson equation, we can rewrite Eq.(3.6) as

$$-\lambda p_i \partial\Sigma_{jk}(p)/\partial\omega + \Lambda_{jik}(p \mid 0, p) = 0 \tag{3.7}$$

which is identical with (2.3) in third-order perturbation theory.

Applying variational differentiation of (3.4) with respect to $\varphi_j^*(2)$, $\varphi_k^*(3)$, we obtain yet another GWI, relating the pair correlator $C$ to the vertex describing the conversion of a quantum with zero wave number and frequency into two quanta (such processes are possible only in higher-order perturbation theory - from third order and up). The corresponding relation in Fourier space is

$$2\lambda p_i \frac{\partial}{\partial\omega}[G_{jj'}^{-1}(p)\, G_{kk'}^{-1}(-p)\, C_{j'k'}(p)] - \Gamma_{jki}(p, -p \mid 0) = 0 \tag{3.8}$$

Relations (3.6) and (3.8) prove useful in investigating questions of compensation for divergences of diagrams in hydrodynamic perturbation theory and elimination of ultraviolet divergences by renormalization.

If one carries out multiplicative renormalizations of the pair correlator, Green's function and vertex by the relations /12/

$$C \to \alpha_1 C, \quad G \to \alpha_2 G, \quad \Gamma \to \alpha_3 \Gamma \tag{3.9}$$

it turns out that such renormalizations are compatible with the Dyson equation provided that $\alpha_1 \alpha_2^2 \alpha_3^2 = 1$. The GWI (3.6) derived from Galilean invariance implies the additional condition $\alpha_2^{-1} = \alpha_3$, which is equivalent to $\alpha_1 = 1$. Thus, invariance under the multiplicative transformations (3.9) is due to the presence of an arbitrary element in the choice of the auxiliary field amplitude $\psi^*$. By introducing a suitable counterterm (not necessarily infinite) and applying a renormalization-group method, which uses the invariance of the result under variations of the normalization point of the field amplitude $\psi^*$, one can determine additional information about the statistical characteristics of the turbulence field, proceeding as in the investigation of turbulent viscosity and diffusion (see, e.g., /13, 14/).

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